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On finite and continuous little groups of representations of semi-simple Lie groups†

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Abstract. The multiplicity of the identity representation occurring in the reduction of representations of semi-simple Lie groups to their finite or continuous subgroups is given for many group–subgroup pairs.

1. Introduction

Recognition of the generality of spontaneous symmetry breaking phenomena in quantum physics is the basis of the interest of physicists in a particular aspect of group theory: orbit structure of representation spaces and the related coarser structure of strata or families of orbits. An extensive bibliography about the subject is to be found in recent reviews (O'Raifeartaigh 1979, Michel 1980, Slansky 1981).

The one-to-one correspondence between strata and little groups reduces the study of strata to that of little groups. Let us recall that H is a little group for a representation $R(\mathrm{G})$ of a compact group G such that $\mathrm{G} \supset \mathrm{H}$, if and only if the reduction

$$R(\mathrm{G}) \supset R(\mathrm{H}) = \bigoplus_i R_i(\mathrm{H}) \tag{1}$$

of $R(\mathrm{G})$ to the representation $R(\mathrm{H})$ of the subgroup H contains the identity representation in the direct sum (1), and H is the largest subgroup of G which leaves invariant the vector forming the basis of this identity representation.

There are two natural questions to ask about little groups.

(i) Given a representation $R(\mathrm{G})$ of a semi-simple Lie group G, what are its little groups?

(ii) Given H and an embedding $\mathrm{H} \subset \mathrm{G}$, which representations $R(\mathrm{G})$ have H as a little group?

The first question is best answered by means of lists of reductions (branching rules) of representations $R(\mathrm{G})$ restricted to all maximal subgroups and then restricted further to maximal subgroups of subgroups, etc such as those of McKay and Patera (1981). An answer to the second question would require an investigation of infinitely many representations $R(\mathrm{G})$ and therefore it has to be given in a different way.

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The purpose of this paper is to answer this second question for many pairs G ⊃ H. In each case G is a semi-simple compact Lie group, while H is either a reductive Lie group (§ 2) or a finite group (§ 4). In § 3 a different problem is considered: given a series of inclusions SU(n) ⊃ H, for $n = n_1, n_2, \ldots$ of a fixed group H in the special unitary groups SU(n), and a representation $\{\mu\}$ of SU(n) specified by a particular Young tableau Y, what are the values of n for which H is a little group of $\{\mu\}$? It turns out that the answer is given in terms of recently found generating functions of a new type (Patera and Sharp 1981). Section 5 contains some closing comments.

2. Generating functions for subgroup scalars. Lie subgroups

Each irreducible representation, Λ, of a semi-simple compact Lie group is labelled by means of the highest weight vector $\Lambda = (a_1, a_2, \ldots, a_k)$, each of whose components is a non-negative integer associated with the projection of Λ in the direction of one of the k simple roots of the corresponding Lie algebra of rank k. These roots are numbered in accordance with the convention of Dynkin (1957, p 116).

It should be noted in particular that throughout this paper an irreducible representation of SU(2), and the locally isomorphic groups Sp(2) and SO(3), is denoted by an integer, a, equal to twice the corresponding angular momentum. In the case of the locally isomorphic groups Sp(4) and SO(5) the labels used, (a_1, a_2), are those appropriate to the Lie algebra C_2. The corresponding label for the Lie algebra B_2 would be (a_2, a_1). Similarly for the groups SU(4) and SO(6) the labels, (a_1, a_2, a_3), are those of A_3. The corresponding D_3 label is (a_2, a_1, a_3). Finally the label for the semi-simple group SO(4) is that of the algebra $A_1 + A_1$ written in the form (a_1, a_2). This is all in accordance with recent convention (McKay and Patera 1981).

The generating function required for subgroup scalars takes the form

$$\mathscr{F}(P_1, P_2, \ldots, P_k) = \sum_{\Lambda} m(\Lambda) P_1^{a_1} P_2^{a_2} \ldots P_k^{a_k} \tag{2}$$

where $m(\Lambda)$ is the number of scalars associated with the restriction of the representation $\Lambda = (a_1, a_2, \ldots, a_k)$ to the appropriate subgroup.

We illustrate this with the generating function for SO(3) scalars in SU(3) representations. Further cases are then listed with only minimal comment.

SU(3) ⊃ SO(3) (Bargmann and Moshinsky 1961)
The generating function is

$$\begin{aligned}\mathscr{F}(P_1, P_2) &= 1/(1-P_1^2)(1-P_2^2)\\ &= 1 + P_1^2 + P_2^2 + P_1^4 + P_1^2P_2^2 + P_2^4 + \ldots + P_1^{2k_1}P_2^{2k_2} \ldots .\end{aligned} \tag{3}$$

Here P_1 and P_2 are auxiliary variables whose exponents, $a_1 = p$ and $a_2 = q$, in the series (3) are the familiar SU(3) irreducible representation labels. Thus a term in the series of the form $m(p, q)P_1^pP_2^q$ means that the SU(3) representation (p, q) of dimension $\frac{1}{2}(p+1)(q+1)(p+q+2)$ contains the scalar representation of SO(3) exactly $m(p, q)$ times. Since SO(3) is a maximal subgroup of SU(3), one immediately sees from (3) that SO(3) is a little group for every SU(3) representation with p and q both even. In each case the multiplicity of the SO(3) scalar is one.

We now gather together a large class of similar results.

$SU(n) \supset SU(n-m) \times SU(m) \times U(1)$, $n \geq 2$ and $1 \leq m \leq [\frac{1}{2}n]$ (Sharp 1972 for $m=2$, Combe *et al* 1979, Patera and Sharp 1980 for $n=6$ and $m=3$)

$$\mathcal{F}(P_1, P_2, \ldots, P_{n-1}) = \left(\prod_{j=1}^{m} (1 - P_j P_{n-j}) \right)^{-1}. \tag{4}$$

Since this embedding is maximal $SU(n-m) \times SU(m) \times U(1)$ is a little group for every $SU(n)$ representation of the form $\Lambda = (a_1, a_2, \ldots, a_{m-1}, a_m, 0, 0, \ldots, 0, a_m, a_{m-1}, \ldots, a_2, a_1)$ if $2m < n$, and of the form $\Lambda = (a_1, a_2, \ldots, a_{m-1}, 2a_m, a_{m-1}, \ldots, a_2, a_1)$ if $2m = n$. In each case the number of scalars is just one.

Let us point out the particular case $n = 5$ and $m = 2$ which underlies the unification of strong, weak and electromagnetic interactions.

$SU(5) \supset SU(3) \times SU(2) \times U(1)$ (Combe *et al* 1979)

$$\mathcal{F}(P_1, P_2, P_3, P_4) = 1/(1 - P_1P_4)(1 - P_2P_3). \tag{4'}$$

In this case the subgroup is the little group for every self-contragredient representation $\Lambda = (a_1, a_2, a_2, a_1)$ of $SU(5)$.

$SU(n) \supset SU(n-m) \times SU(m)$, $n \geq 2$ and $1 \leq m \leq [\frac{1}{2}n]$ (Sharp 1972 for $m=2$, Combe *et al* 1979, Patera and Sharp 1980 for $n=6$ and $m=3$)
This is the same as the previous case except that the U(1) content is now ignored. The generating function is different however:

$$\mathcal{F}(P_1, P_2, \ldots, P_{n-1}) = 1/(1 - P_m)(1 - P_{n-m}) \prod_{j=1}^{m-1} (1 - P_j P_{n-j}). \tag{5}$$

Scalars occur in the reduction of the representations $\Lambda = (a_1, a_2, \ldots, a_{m-1}, a_m, 0, 0, \ldots, 0, a_{n-m}, a_{m-1}, \ldots, a_2, a_1)$ if $2m < n$, and $\Lambda = (a_1, a_2, \ldots, a_{m-1}, a_m, a_{m-1}, \ldots, a_2, a_1)$ if $2m = n$. The number of scalars in the first case is just one, but in the second it is $a_m + 1$ by virtue of the generating function containing the factor $1/(1 - P_m)^2$. This embedding is not maximal, but $SU(n-m) \times SU(m)$ is the required little group for the appropriate representations of $SU(n)$ if $2m < n$ and $a_m \neq a_{n-m}$, or if $2m = n$ and $a_m \neq 0$.

$SU(n) \supset SU(n-1) \times U(1)$, $n \geq 2$ (Weyl 1931, p 391, Gel'fand and Zetlin 1950, Sharp and Lam 1969)

$$\mathcal{F}(P_1, P_2, \ldots, P_{n-1}) = 1/(1 - P_1 P_{n-1}). \tag{6}$$

This is the special case of (4) corresponding to $m = 1$, where SU(1) consists of the identity element alone. Further specialisation to $n = 2$ gives:

$SU(2) \supset U(1)$ and $SU(2) \supset SO(2)$ cf (8) and $SO(3) \supset SO(2)$ cf (11)

$$\mathcal{F}(P_1) = 1/(1 - P_1^2). \tag{6'}$$

$SU(n) \supset SU(n-1)$, $n \geq 2$ (Weyl 1931, p 391, Gel'fand and Zetlin 1950)

$$\mathcal{F}(P_1, P_2, \ldots, P_{n-1}) = 1/(1 - P_1)(1 - P_{n-1}). \tag{7}$$

This is the special case of (5) with $m = 1$. Further specialisation to $n = 2$ only yields the generating function for the dimension, $a + 1$, of the representation $\Lambda = (a)$ of SU(2).

SU(n) ⊃ SO(n), $n \geqslant 2$ (Littlewood 1950, p 240, Bargmann and Moshinsky 1961 for $n=3$, Sharp and Lam 1969 for $n=4$, Combe *et al* 1979, Patera and Sharp 1981 for $n=5$)

$$\mathscr{F}(P_1, P_2, \ldots, P_{n-1}) = \left(\prod_{j=1}^{n-1} (1-P_j^2)\right)^{-1}. \tag{8}$$

Clearly (3) is the particular case $n=3$ of (8), and (6′) is the case $n=2$. In general SO(n) is a little group for every SU(n) representation with all labels even: $\Lambda = (2a_1, 2a_2, \ldots, 2a_{n-1})$. This merely corresponds to the fact that the associated group character $\{\lambda\}$ is such that $\lambda = \delta$ with δ in the series D, so that λ is a partition all of whose parts are even (Littlewood 1950, p 240, King 1975).

SU($2k$) ⊃ Sp($2k$), $k \geqslant 2$ (Littlewood 1950, p 295, Sharp 1970 for $k=2$, Combe *et al* 1979, Couture and Sharp 1980 for $k=3$)

$$\mathscr{F}(P_1, P_2, \ldots, P_{2k-1}) = \left(\prod_{j=1}^{k-1} (1-P_{2j})\right)^{-1}. \tag{9}$$

Thus Sp($2k$) is a little group for every representation of SU($2k$) with a label of the form $\Lambda = (0, a_2, 0, a_4, \ldots, a_{2k-2}, 0)$. This corresponds to the fact that the associated group character $\{\lambda\}$ is such that $\lambda = \beta$ with β in the series B, conjugate to D, so that λ is a partition with each part repeated an even number of times (Littlewood 1950, p 295, King 1975).

SO($2k+1$) ⊃ SO($2k+1-m$) × SO(m), $k \geqslant 1$, $k \neq 2$ and $1 \leqslant m \leqslant k$ (King 1975, Patera *et al* 1980 for $k=3$ and $m=2$)

$$\mathscr{F}(P_1, P_2, \ldots, P_k) = \begin{cases} \left((1-P_m)\prod_{j=1}^{m-1} (1-P_j^2)\right)^{-1} & \text{for } 1 \leqslant m \leqslant k-1 \\ \left(\prod_{j=1}^{m} (1-P_j^2)\right)^{-1} & \text{for } m = k. \end{cases} \tag{10}$$

Thus the tensor representation $[\lambda]$ of SO($2k+1$) yields a scalar if and only if λ is a partition into no more than m parts such that either $\lambda = \delta$ or $\lambda/1^m = \delta$, for some δ in the series D.

SO($2k+1$) ⊃ SO($2k$), $k \geqslant 1$, $k \neq 2$ (Boerner 1963, p 251)

$$\mathscr{F}(P_1, P_2, \ldots, P_k) = \begin{cases} 1/(1-P_1) & \text{for } k \geqslant 3 \\ 1/(1-P_1^2) & \text{for } k = 1. \end{cases} \tag{11}$$

This result follows from (10) with $m=1$ since SO(1) is merely the identity element.

SO($2k+1$) ⊃ SO($2k-1$), $k \geqslant 1$, $k \neq 2$ (Patera *et al* 1980)

$$\mathscr{F}(P_1, P_2, \ldots, P_k) = \begin{cases} 1/(1-P_2)(1-P_1)^2 & \text{for } k \geqslant 3 \\ 1/(1-P_1)^2 & \text{for } k = 1. \end{cases} \tag{12}$$

This embedding is not maximal so that it is necessary to compare with (11) in determining whether or not SO($2k-1$) is a little group for a representation of SO($2k+1$).

The results (10), (11) and (12) are not appropriate to the case $k=2$ simply because of the labelling adopted (McKay and Patera 1981) for representations of SO(5). It is only necessary to interchange P_1 and P_2 in these formulae to obtain:

$\mathrm{SO}(5) \supset \mathrm{SO}(3) \times \mathrm{SO}(2)$ and $\mathrm{Sp}(4) \supset \mathrm{SU}(2) \times \mathrm{U}(1)$ cf (18)

$$\mathscr{F}(P_1, P_2) = 1/(1-P_1^2)(1-P_2^2). \tag{10$'$}$$

$\mathrm{SO}(5) \supset \mathrm{SO}(4)$ and $\mathrm{Sp}(4) \supset \mathrm{Sp}(2) \times \mathrm{Sp}(2)$ cf (15) (Malkin and Mandel'tsveig 1966, Couture and Sharp 1980)

$$\mathscr{F}(P_1, P_2) = 1/(1-P_2). \tag{11$'$}$$

$\mathrm{SO}(5) \supset \mathrm{SO}(3)$ and $\mathrm{Sp}(4) \supset \mathrm{SU}(2)$ cf (19)

$$\mathscr{F}(P_1, P_2) = 1/(1-P_1^2)(1-P_2)^2. \tag{12$'$}$$

$\mathrm{SO}(2k+1) \supset \mathrm{SU}(k) \times \mathrm{U}(1)$, $k \geqslant 1$, $k \neq 2$ (King 1975)

$$\mathscr{F}(P_1, P_2, \ldots, P_k) = \left((1-P_k^2) \prod_{j=1}^{k-1} (1-P_j)\right)^{-1}. \tag{13}$$

Thus every tensor representation $[\lambda]$ yields a scalar. However this embedding is not maximal so that comparison must be made with $\mathrm{SO}(2k+1) \supset \mathrm{SO}(2k)$ and $\mathrm{SO}(2k) \supset \mathrm{SU}(k) \times \mathrm{U}(1)$ in determining the true little group.

$\mathrm{SO}(2k+1) \supset \mathrm{SU}(k)$, $k \geqslant 1$, $k \neq 2$

$$\mathscr{F}(P_1, P_2, \ldots, P_k) = \left((1-P_k)^2 \prod_{j=1}^{k-1} (1-P_j)\right)^{-1}. \tag{14}$$

Dropping the dependence on U(1) in (13) gives rise to a_k+1 scalars in the case of the tensor representations $[\lambda]$ for which $a_k = 2\lambda_k$ and the spinor representations $[\Delta; \lambda]$ for which $a_k = 2\lambda_k + 1$.

The $k=2$ case involves interchanging P_1 and P_2 as before.

$\mathrm{SO}(5) \supset \mathrm{SU}(2) \times \mathrm{U}(1)$ (Malkin and Mandel'tsveig 1966)

$$\mathscr{F}(P_1, P_2) = 1/(1-P_1^2)(1-P_2). \tag{13$'$}$$

$\mathrm{SO}(5) \supset \mathrm{SU}(2)$ (Malkin and Mandel'tsveig 1966)

$$\mathscr{F}(P_1, P_2) = 1/(1-P_1)^2(1-P_2). \tag{14$'$}$$

$\mathrm{Sp}(2k) \supset \mathrm{Sp}(2k-2m) \times \mathrm{Sp}(2m)$, $k \geqslant 2$ and $1 \leqslant m \leqslant [\frac{1}{2}k]$ (Sharp 1970 for $m=1$, King 1975)

$$\mathscr{F}(P_1, P_2, \ldots, P_k) = \left(\prod_{j=1}^{m} (1-P_{2j})\right)^{-1}. \tag{15}$$

This corresponds to the statement that the representation $\langle\lambda\rangle$ yields a scalar if and only if λ is a partition into no more than m parts and $\lambda=\beta$ with β in the series B, as in the case (9).

$\mathrm{Sp}(2k)\supset \mathrm{Sp}(2k-2)\times \mathrm{U}(1)$, $k\geqslant 2$ (Zhelobenko 1962, King 1976)

$$\mathscr{F}(P_1,P_2,\ldots,P_k)=1/(1-P_2)(1-P_1^2). \tag{16}$$

$\mathrm{Sp}(2k)\supset \mathrm{Sp}(2k-2)$, $k\geqslant 2$ (Miller 1966, Hegerfeldt 1966)

$$\mathscr{F}(P_1,P_2,\ldots,P_k)=1/(1-P_2)(1-P_1)^2. \tag{17}$$

$\mathrm{Sp}(2k)\supset \mathrm{SU}(k)\times \mathrm{U}(1)$, $k\geqslant 1$ (Malkin and Mandel'tsveig 1966 for $k=2$, Sharp and Lam 1969 for $k=2$, King 1975, Gaskell *et al* 1981 for $k=3$)

$$\mathscr{F}(P_1,P_2,\ldots,P_k)=\left(\prod_{j=1}^{k}(1-P_j^2)\right)^{-1}. \tag{18}$$

Thus the representation $\langle\lambda\rangle$ yields a scalar if and only if $\lambda=\delta$ with δ in the series D, as in (8).

$\mathrm{Sp}(2k)\supset \mathrm{SU}(k)$, $k\geqslant 1$ (Malkin and Mandel'tsveig 1966 for $k=2$)

$$\mathscr{F}(P_1,P_2,\ldots,P_k)=\left((1-P_k)^2\prod_{j=1}^{k-1}(1-P_j^2)\right)^{-1}. \tag{19}$$

$\mathrm{SO}(k)\supset \mathrm{SO}(2k-m)\times \mathrm{SO}(m)$, $k\geqslant 2$, $k\neq 3$ and $1\leqslant m\leqslant k$ (King 1975)

$$\mathscr{F}(P_1,P_2,\ldots,P_k)=\begin{cases}\left((1-P_m)\prod_{j=1}^{m-1}(1-P_j^2)\right)^{-1} & \text{for } 1\leqslant m\leqslant k-2\\[2ex] \left((1-P_mP_{m+1})\prod_{j=1}^{m-1}(1-P_j^2)\right)^{-1} & \text{for } m=k-1\\[2ex] \left(\prod_{j=1}^{m}(1-P_j^2)\right)^{-1} & \text{for } m=k.\end{cases} \tag{20}$$

As in the case of (10) this corresponds to a scalar appearing in each tensor representation $[\lambda]$, $[\lambda]_+$ or $[\lambda]_-$ with λ a partition into no more than m parts and either $\lambda=\delta$ or $\lambda/1^m=\delta$ for some δ in the series D.

$\mathrm{SO}(2k)\supset \mathrm{SO}(2k-1)$, $k\geqslant 2$, $k\neq 3$ (Boerner 1963, p 253)

$$\mathscr{F}(P_1,P_2,\ldots,P_k)=\begin{cases}1/(1-P_1) & \text{for } k\geqslant 4\\ 1/(1-P_1P_2) & \text{for } k=2.\end{cases} \tag{21}$$

This result is the special case of (20) with $m=1$.

$\mathrm{SO}(2k)\supset \mathrm{SO}(2k-2)$, $k\geqslant 2$, $k\neq 3$ (King 1976)

$$\mathscr{F}(P_1,P_2,\ldots,P_k)=\begin{cases}1/(1-P_2)(1-P_1)^2 & \text{for } k\geqslant 4\\[1ex] \dfrac{(1+P_1P_2)}{(1-P_1P_2)(1-P_1^2)(1-P_2^2)} & \text{for } k=2.\end{cases} \tag{22}$$

The results appropriate to $k=3$ may be recovered by interchanging P_1 and P_2.

$SO(6)\supset SO(3)\times SO(3)$ and $SU(4)\supset SO(4)$ cf (8)

$$\mathscr{F}(P_1,P_2,P_3)=1/(1-P_1^2)(1-P_2^2)(1-P_3^2). \tag{20'}$$

$SO(6)\supset SO(4)\times SO(2)$ and $SU(4)\supset SU(2)\times SU(2)\times U(1)$ cf (4)

$$\mathscr{F}(P_1,P_2,P_3)=1/(1-P_1P_3)(1-P_2^2). \tag{20''}$$

$SO(6)\supset SO(5)$ and $SU(4)\supset Sp(4)$ cf (9)

$$\mathscr{F}(P_1,P_2,P_3)=1/(1-P_2). \tag{21'}$$

$SO(6)\supset SO(4)$ and $SU(4)\supset SU(2)\times SU(2)$ cf (5)

$$\mathscr{F}(P_1,P_2,P_3)=1/(1-P_1P_3)(1-P_2)^2. \tag{22'}$$

$SO(4m+2)\supset SU(2m+1)\times U(1)$, $m\geqslant 2$

$$\mathscr{F}(P_1,P_2,\ldots,P_{2m+1})=\left((1-P_{2m}P_{2m+1})\prod_{j=1}^{m-1}(1-P_{2j})\right)^{-1}. \tag{23}$$

$SO(4m+2)\supset SU(2m+1)$, $m\geqslant 2$

$$\mathscr{F}(P_1,P_2,\ldots,P_{2m+1})=\left((1-P_{2m-1})\prod_{j=1}^{m}(1-P_{2j})\right)^{-1}. \tag{24}$$

To recover the results for SO(6) it is merely necessary to set $m=1$ in (23) and (24) and to interchange P_1 and P_2, yielding:

$SO(6)\supset SU(3)\times U(1)$ and $SU(4)\supset SU(3)\times U(1)$ cf (4)

$$\mathscr{F}(P_1,P_2,P_3)=1/(1-P_1P_3). \tag{23'}$$

$SO(6)\supset SU(3)$ and $SU(4)\supset SU(3)$ cf (5)

$$\mathscr{F}(P_1,P_2,P_3)=1/(1-P_1)(1-P_3). \tag{24'}$$

$SO(4m)\supset SU(2m)\times U(1)$, $m\geqslant 2$

$$\mathscr{F}(P_1,P_2,\ldots,P_{2m})=\left((1-P_{2m-1}^2)\prod_{j=1}^{m-1}(1-P_{2j})\right)^{-1}. \tag{25}$$

$SO(4m)\supset SU(2m)$, $m\geqslant 2$

$$\mathscr{F}(P_1,P_2,\ldots,P_{2m})=\left((1-P_{2m-1})^2\prod_{j=1}^{m-1}(1-P_{2j})\right)^{-1}. \tag{26}$$

It is possible to apply an outer automorphism to $SO(2k)$ which is equivalent to interchanging P_k and P_{k-1} as far as subgroup scalar generating functions are concerned. This gives nothing new in (23) and (24), but it does in (25) and (26). The new results are obtained by replacing P_{2m-1} by P_{2m}. More significantly in the special case of SO(8) the group of outer automorphisms is of order 6, and is equivalent to arbitrary permutations of P_1, P_3 and P_4. Interchanging P_1 and P_3 in (25) and (26) in the case $m=2$ then gives:

$\mathrm{SO}(8)\supset\mathrm{SU}(4)\times\mathrm{U}(1)$ and $\mathrm{SO}(8)\supset\mathrm{SO}(6)\times\mathrm{SO}(2)$ cf (20)

$$\mathscr{F}(P_1,P_2,P_3,P_4)=1/(1-P_1^2)(1-P_2). \tag{25'}$$

$\mathrm{SO}(8)\supset\mathrm{SU}(4)$ and $\mathrm{SO}(8)\supset\mathrm{SO}(6)$ cf (22)

$$\mathscr{F}(P_1,P_2,P_3,P_4)=1/(1-P_1)^2(1-P_2). \tag{26'}$$

In addition to these results for the classical groups the following apply to the exceptional groups:

$\mathrm{SO}(7)\supset\mathrm{G}(2)$ (Wybourne 1972, Gaskell *et al* 1978)

$$\mathscr{F}(P_1,P_2,P_3)=1/(1-P_3). \tag{27}$$

$\mathrm{G}(2)\supset\mathrm{SU}(3)$ (Fronsdal 1962, Mandel'tsveig 1965, Sharp and Lam 1969, Gaskell *et al* 1978)

$$\mathscr{F}(P_1,P_2)=1/(1-P_2). \tag{28}$$

$\mathrm{G}(2)\supset\mathrm{SU}(2)\times\mathrm{SU}(2)$ (Stone 1970, Gaskell and Sharp 1981)

$$\mathscr{F}(P_1,P_2)=1/(1-P_1^2)(1-P_2^2). \tag{29}$$

$\mathrm{F}(4)\supset\mathrm{SO}(9)$ (Wybourne and Bowick 1977)

$$\mathscr{F}(P_1,P_2,P_3,P_4)=1/(1-P_4). \tag{30}$$

3. Fixed plethysm generating functions

Examination of the tables (McKay and Patera 1981) of branching rules for representations of simple Lie groups restricted to maximal semi-simple Lie groups shows that the generating functions of the previous section cover 106 embeddings involving the classical groups and 4 embeddings involving the exceptional groups.

There are other maximal embeddings of the form $\mathrm{SU}(nm)\supset\mathrm{SU}(n)\times\mathrm{SU}(m)$ which are covered by the work of Combe *et al* (1979). However in these cases no simple generating functions for the subgroup scalars exist. The same is true of the maximal embeddings of the form $\mathrm{SO}(nm)\supset\mathrm{SO}(n)\times\mathrm{SO}(m)$, $\mathrm{SO}(4nm)\supset\mathrm{Sp}(2n)\times\mathrm{Sp}(2m)$ and $\mathrm{Sp}(2nm)\supset\mathrm{Sp}(2n)\times\mathrm{SO}(m)$. This is a consequence of the complexity of the branching rules given elsewhere (King 1975) for these embeddings.

The remaining embeddings covered by the tables are all associated with plethysms and include 13 involving the embedding of SU(2) or SO(3) in a simple Lie group. First we consider the inclusion $\mathrm{SU}(2)\subset\mathrm{SU}(n)$, where the n-dimensional defining representation of SU(n) contains the irreducible representation $\Lambda=(n-1)$ of SU(2). This inclusion of SU(2) in SU(n) is unique and SU(2) is then called the principal three-dimensional subgroup.

For $n \geq 4$ this inclusion is not maximal however, since for n even we have $\mathrm{SU}(n) \supset \mathrm{Sp}(n) \supset \mathrm{SU}(2)$ and for n odd $\mathrm{SU}(n) \supset \mathrm{SO}(n) \supset \mathrm{SO}(3)$, whilst for the particular case $n = 7$ $\mathrm{SU}(7) \supset \mathrm{SO}(7) \supset \mathrm{G}(2) \supset \mathrm{SO}(3)$.

The cases $n = 4$ and 5 coincide. The generating function for subgroup scalars takes the form:

$\mathrm{Sp}(4) \supset \mathrm{SU}(2)$ and $\mathrm{SO}(5) \supset \mathrm{SO}(3)$ (Stone 1970, Gaskell *et al* 1978)

$$\mathscr{F}(P_1, P_2) = (1 + P_1^6 P_2^3)/(1 - P_1^4)(1 - P_2^3)(1 - P_1^4 P_2^2). \tag{31}$$

For $n = 7$ we have:

$\mathrm{G}(2) \supset \mathrm{SO}(3)$ (Gaskell and Sharp 1981)

$$\begin{aligned}
\mathscr{F}(P_1, P_2) = {} & [(1 - P_1^2)(1 - P_2^6)(1 - P_1^2 P_2^2)]^{-1}\{[(1 + P_1^2 P_2^5)(1 + P_2^{15}) + P_1 P_2^7 (1 + P_2)(1 + P_2^5)] \\
& \times [(1 - P_2^4)(1 - P_2^{10})]^{-1} + (1 + P_2^5)[P_1(P_2^5 + P_2^9) + P_1^2(P_2^4 + P_2^7 + P_2^8 + P_2^{11}) \\
& + P_1^3(P_2 + P_2^4 + P_2^5 + P_2^8) + P_1^4(P_2^3 + P_2^6 + P_2^7 + P_2^{10}) + P_1^5(P_2^4 + P_2^5 + P_2^7 + P_2^8) \\
& + P_1^6(P_2 + P_2^4 + P_2^7 + P_2^{10}) + P_1^7(P_2^3 + P_2^4 + P_2^6 + P_2^7) + P_1^8(P_2 + P_2^4 + P_2^5 + P_2^8) \\
& + P_1^9(P_2^3 + P_2^6 + P_2^7 + P_2^{10}) + P_1^{10}(1 + P_2^3 + P_2^4 + P_2^7) \\
& + P_1^{11}(P_2^2 + P_2^6)][(1 - P_2^{10})(1 - P_1^{10})]^{-1} \\
& + (1 + P_2^3)[P_1^6 + P_1^{15} + (P_1^9 + P_1^{11} + P_1^{12} + P_1^{14})P_2 + (P_1^4 + P_1^6 + P_1^7 + P_1^9)P_2^2 \\
& + (P_1^3 + P_1^{12})P_2^3][(1 - P_1^{10})(1 - P_1^6)]^{-1}\}.
\end{aligned} \tag{32}$$

This is the inclusion where the 7-dimensional irreducible representation (0, 1) of G(2) contains the 7-dimensional irreducible representation (6) of SO(3).

This rapid increase in complexity as n increases makes further progress in dealing with these maximal subgroups unlikely. However in order to complete what is known about generating functions for continuous subgroup scalars we now turn to a recently discovered type of generating function (Patera and Sharp 1981). Particular cases give us generating functions for subgroup scalars. However, the interpretation and use of these functions is quite different from the previous ones. In all cases of § 2 (and also in all cases in the following § 4) the group and subgroup are fixed and each term of the generating function power series refers to a different irreducible representation of the containing group. From now on in this section the larger group is always of the type SU(n) but *is not fixed*; its representation is specified by a Young tableau and *is fixed*. Each term of the corresponding generating function power series refers to a *different* group SU(n) but to the *same* Young tableau.

This new generating function takes the form

$$\mathscr{G}_{\{\mu\}}(L_1, L_2, \ldots, L_k) = \sum_{\Lambda} m(\Lambda) L_1^{a_1} L_2^{a_2} \ldots L_k^{a_k} \tag{33}$$

where now $m(\Lambda)$ is the number of scalars associated with the restriction of the representation of SU(n) specified by the Young tableau $\{\mu\}$ to a semi-simple Lie subgroup H of rank k, possessing an n-dimensional representation labelled in the usual way by $\Lambda = (a_1, a_2, \ldots, a_k)$.

If the subgroup H is SU(2) or SO(3) we have

$$\mathcal{G}_{\{\mu\}}(L)=\sum_{a} m(a)L^{a} \tag{33'}$$

and typically (Patera and Sharp 1981)

$$\mathcal{G}_{\{1\}}(L)=1 \tag{34}$$

$$\mathcal{G}_{\{2\}}(L)=1/(1-L^{2}) \tag{35}$$

$$\mathcal{G}_{\{3\}}(L)=1/(1-L^{4}) \tag{36}$$

$$\mathcal{G}_{\{4\}}(L)=1/(1-L^{2})(1-L^{3}) \tag{37}$$

$$\mathcal{G}_{\{5\}}(L)=(1+L^{18})/(1-L^{4})(1-L^{8})(1-L^{12}). \tag{38}$$

Similarly

$$\mathcal{G}_{\{1^2\}}(L)=L/(1-L^{2}) \tag{39}$$

$$\mathcal{G}_{\{1^3\}}(L)=L^{2}/(1-L^{4}) \tag{40}$$

and generally

$$\mathcal{G}_{\{1^p\}}(L)=L^{p-1}\mathcal{G}_{\{p\}}(L). \tag{41}$$

Also

$$\mathcal{G}_{\{2^2\}}(L)=L/(1-L)(1-L^{3}) \tag{42}$$

$$\begin{aligned}\mathcal{G}_{\{2^3\}}(L)&=L\mathcal{G}_{\{3^2\}}(L)\\&=L^{2}(1+L^{9})/(1-L^{2})^{2}(1-L^{4})(1-L^{6}).\end{aligned} \tag{43}$$

Each function is a power series in L where the exponent, a, indicates that the relevant group is SU(n) with $n=a+1$.

To be specific consider (42) for $n\geqslant 2$ with the SU(n) representation denoted either in the Young tableau, partition notation by $\{2^2\}$ or in the Dynkin convention by $(0, 2, 0, \ldots, 0)$. One has

$$\mathcal{G}_{\{2^2\}}(L)=L+L^{2}+L^{3}+2L^{4}+\ldots. \tag{42'}$$

The first term is trivial because the representation $\{2^2\}$ of SU(2) is of dimension 1 and is the identity representation. The second term corresponds to the fact that the representation $\{2^2\}=(0,2)$ of SU(3) gives rise to a single scalar of SO(3) on restriction in accordance with (8). The third and fourth terms correspond to the representations $\{2^2\}=(0,2,0)$ of SU(4) and $\{2^2\}=(0,2,0,0)$ of SU(5) containing one and two scalars of SU(2) respectively.

The generating functions

$$\mathcal{G}_{\{2,1\}}(L)=\mathcal{G}_{\{2,1^2\}}(L)=\mathcal{G}_{\{3,1\}}(L)=0 \tag{44}$$

indicate merely that the corresponding SU(n) representations do not contain SU(2) scalars for any value of n.

Some remarkable symmetries of fixed plethysm generating functions have been pointed out (Patera and Sharp 1981); namely, the multiplicities of SU(2) scalars coincide in certain representations of infinite series of different SU(n) groups. More precisely, it happens for the representations

$$\{x, 1^{y-1}\} \text{ of } \mathrm{SU}(n+y) \text{ and } \{y, 1^{x-1}\} \text{ of } \mathrm{SU}(n+x) \tag{45}$$

for any n, and for the six representations obtained from

$$\{x^y\} \text{ of } \mathrm{SU}(y+z) \tag{46}$$

by permuting x, y and z in all possible ways. Here the integers x, y and z are non-negative and such that the symbols are meaningful. Notice that (41) is a particular case of both (45) and (46). The generating functions for the general cases (45) and (46) are not known.

In the following case the subgroup of SU(n) is SU(3). It is embedded in SU(n) in such a way that the SU(3) representation (p, q) is contained irreducibly in the defining representation of SU(n). Therefore one must have the equality of dimensions:

$$n = \tfrac{1}{2}(p+1)(q+1)(p+q+2). \tag{47}$$

Then, for example

$$\begin{aligned}\mathscr{G}_{\{2\}}(L_1, L_2) &= 1/(1-L_1L_2)\\ &= 1 + L_1L_2 + \ldots + L_1^pL_2^p + \ldots\end{aligned} \tag{48}$$

that is, in order to find SU(3) scalars in the SU(n) representation $(2, 0, \ldots, 0)$, it is necessary that n is given by (47) and that $p = q$. However even in these cases there is only one SU(3) scalar and it is easy to see that the corresponding little group is actually SO(n), since if $p = q$ the representation matrices of (p, p) are orthogonal and the representation $(2, 0, \ldots, 0)$ gives rise to a scalar of SO(n) by virtue of (8).

In addition one finds (Patera and Sharp 1981, equation (5.6))

$$\mathscr{G}_{\{1^2\}}(L_1, L_2) = 1$$

indicating that for no value of n does the representation $(0, 1, \ldots, 0)$ of SU(n) contain an SU(3) scalar.

Finally, the generating functions for scalars of the groups SU(2) embedded into SU(n) as a direct sum $a_1 \oplus a_2$ of two arbitrary irreducible representations of dimensions $a_1 + 1$ and $a_2 + 1$, such that $a_1 + a_2 + 2 = n$, are given by:

$$\mathscr{G}_{\{2\}}(L_1, L_2) = \frac{1}{(1-L_1)(1-L_2^2)} + \frac{1}{(1-L_1L_2)} + \frac{1}{(1-L_1^2)(1-L_2)} \tag{49}$$

$$\mathscr{G}_{\{1^2\}}(L_1, L_2) = \frac{L_2}{(1-L_1)(1-L_2^2)} + \frac{1}{(1-L_1L_2)} + \frac{L_1}{(1-L_1^2)(1-L_2)} \tag{50}$$

$$\mathscr{G}_{\{3\}}(L_1, L_2) = \frac{1}{(1-L_1)(1-L_2^4)} + \frac{1}{(1-L_2^2)(1-L_1^2L_2)} + \{L_1 \leftrightarrow L_2\} \tag{51}$$

$$\mathscr{G}_{\{1^3\}}(L_1, L_2) = \frac{L_2^2}{(1-L_1)(1-L_2^4)} + \frac{L_2}{(1-L_2^2)(1-L_1^2L_2)} + \{L_1 \leftrightarrow L_2\} \tag{52}$$

$$\mathscr{G}_{\{2,1\}}(L_1, L_2) = \frac{1}{(1-L_2^2)(1-L_1^2L_2)} + \frac{1}{(1-L_1^2)(1-L_1L_2^2)} \tag{53}$$

The variables L_1 and L_2 carry the representation labels a_1 and a_2 respectively as exponents.

4. Generating functions for subgroup scalars. Finite subgroups

In this section we collect the generating functions for little groups of SU(2) representations (Patera *et al* 1978, Desmier and Sharp 1979) and give a new one for the subgroup of SU(3) of order 168:

$SU(2) \supset {}^{(d)}T$

$$\mathscr{F}(P) = \frac{1+P^{12}}{(1-P^6)(1-P^8)} = 1 + P^6 + P^8 + 2P^{12} + \dots \tag{54}$$

Here the presence of the term $2P^{12}$ means that the irreducible SU(2) representation of dimension 13 contains two scalars of the double tetrahedral group ${}^{(d)}T$ in the direct sum (1).

$SU(2) \supset {}^{(d)}O$

$$\mathscr{F}(P) = \frac{1+P^{18}}{(1-P^8)(1-P^{12})} \tag{55}$$

$SU(2) \supset {}^{(d)}I$

$$\mathscr{F}(P) = \frac{1+P^{30}}{(1-P^{12})(1-P^{20})} \tag{56}$$

$SU(2) \supset {}^{(d)}D_n$, $n = 1, 2, \dots$

$$\mathscr{F}(P) = \frac{1+P^{2n+2}}{(1-P^4)(1-P^{2n})}. \tag{57}$$

For $n = 1$, the group consists of two elements ± 1.

$SU(2) \supset {}^{(d)}C_n$, $n = 1, 2, \dots$

$$\mathscr{F}(P) = \frac{1+P^{2n}}{(1-P^2)(1-P^{2n})} \tag{58}$$

The subsequent groups occur only in O(3).

$O(3) \supset T[O$

$$\mathscr{F}(P) = \frac{1}{(1-P^6)(1-P^8)} \tag{59}$$

$O(3) \supset D_n[D_{2n}$, $n = 1, 2, \dots$

$$\mathscr{F}(P) = \frac{1+P^{2n+2}}{(1-P^4)(1-P^{4n})} \qquad \text{if } n \text{ is even} \tag{60}$$

$$\mathscr{F}(P) = \frac{1}{(1-P^4)(1-P^{2n})} \qquad \text{if } n \text{ is odd} \tag{61}$$

$O(3) \supset C_n[D_n, n = 1, 2, \ldots$

$$\mathscr{F}(P) = \frac{1}{(1-P^2)(1-P^{2n})}. \tag{62}$$

$O(3) \supset C_n[C_{2n}, n = 1, 2, \ldots$

$$\mathscr{F}(P) = \frac{1+2P^{2n+2}+P^{4n}}{(1-P^4)(1-P^{4n})} \qquad \text{if } n \text{ is even} \tag{63}$$

$$\mathscr{F}(P) = \frac{1+P^{2n}}{(1-P^4)(1-P^{2n})} \qquad \text{if } n \text{ is odd.} \tag{64}$$

To conclude the section let us consider a finite subgroup $\Sigma(168)$ of SU(3) which is not a subgroup of O(3). Its order is 168. Details concerning $\Sigma(168)$ and its inclusion in SU(3) were studied by Fairbairn *et al* (1964).

Standard methods (for instance, Patera *et al* 1978) applied in a straightforward way provide (Desmier *et al* 1981) the generating function for $\Sigma(168)$ scalars in reduction of the irreducible representations of SU(3):

$$\begin{aligned}
\mathscr{F}(P_1, P_2) = {} & \frac{1}{(1-P_1^{14})(1-P_2^{14})}\Big(\frac{1}{(1-P_1^4)(1-P_1^6)(1-P_2^6)} \\
& \times[1+P_2^4+P_1(P_2^9+P_2^{13}+P_2^{14}+P_2^{17}+P_2^{18}) \\
& +P_1^2P_2^{16}+P_1^3(P_2^9+P_2^{11}+P_2^{12}+P_2^{13}+P_2^{14}+2P_2^{15}+P_2^{16}) \\
& +P_1^4(P_2^{13}+P_2^{15})+P_1^5(P_2^{10}+P_2^{11}+P_2^{13}+P_2^{15}+P_2^{17}) \\
& +P_1^6P_2^{13}+P_1^7(P_2^7+P_2^{11})+P_1^{14}(P_2^{14}+P_2^{18}) \\
& +P_1^{15}(P_2^7+P_2^{11})+P_1^{16}P_2^5+P_1^{17}(P_2+P_2^5+P_2^8)+P_1^{18}(P_2^3+P_2^5) \\
& +P_1^{19}(P_2^2+2P_2^3+P_2^4+P_2^6+P_2^7)+P_1^{20}P_2^2+P_1^{21}(1+P_2+P_2^4+P_2^5+P_2^9)] \\
& +\frac{1}{(1-P_1^6)(1-P_2^4)(1-P_2^6)}[P_2^8+P_2^{21}+P_1(P_2^{17}+P_2^{21})+P_1^2(P_2^{19}+P_2^{20}) \\
& +P_1^3(P_2^{17}+P_2^{18}+2P_2^{19})+P_1^4(P_2^8+P_2^{19}+P_2^{21}) \\
& +P_1^5(P_2^{16}+P_2^{17}+P_2^{18}+P_2^{19}+P_2^{21})+P_1^6P_2^{19}+P_1^7(P_2^7+P_2^{15}+P_2^{17}+P_2^{19}) \\
& +P_1^8P_2^{17}+P_1^9(P_2+P_2^{19}+P_2^{21})+P_1^{10}P_2^5+P_1^{11}(P_2^3+P_2^7+P_2^{15}) \\
& +P_1^{12}P_2^3+P_1^{13}(P_2+P_2^4+P_2^5+P_2^6)+P_1^{14}(P_2+P_2^3+P_2^{22}) \\
& +P_1^{15}(2P_2^3+P_2^4)+P_1^{16}(P_2^2+P_2^3)+P_1^{17}(P_2+P_2^5)+P_1^{18}(P_2+P_2^{22})] \\
& +\frac{1}{(1-P_1^4)(1-P_2^4)(1-P_2^6)}[P_1(P_2^{16}+P_2^{18}) \\
& +P_1^2(P_2^{15}+P_2^{17}+P_2^{18}+P_2^{20})+P_1^3(P_2^{10}+P_2^{12}+P_2^{19}+P_2^{20}) \\
& +P_1^4(P_2^2+P_2^5+P_2^{16}+2P_2^{17}+P_2^{18}+P_2^{20}) \\
& +P_1^5(P_2^3+P_2^{13}+P_2^{14}+P_2^{20}+P_2^{21}) \\
& +P_1^6(P_2^2+P_2^3+P_2^5+P_2^{16}+P_2^{17}+P_2^{18}+P_2^{20}) \\
& +P_1^7(P_2^4+P_2^5+P_2^6+P_2^{14}+P_2^{16}+P_2^{18})
\end{aligned}$$

$$+P_1^8(P_2+P_2^2+P_2^3+P_2^4+P_2^{18}+P_2^{19}+P_2^{20}+P_2^{21})$$
$$+P_1^9(P_2^4+P_2^6+P_2^8+P_2^{16}+P_2^{17}+P_2^{18})$$
$$+P_1^{10}(P_2^2+P_2^4+P_2^5+P_2^6+P_2^{17}+P_2^{19}+P_2^{20}$$
$$+P_1^{11}(P_2+P_2^2+P_2^8+P_2^9+P_2^{17}+P_2^{19})$$
$$+P_1^{12}(P_2^2+P_2^4+2P_2^5+P_2^6+P_2^{17}+P_2^{20})$$
$$+P_1^{13}(P_2^2+P_2^3+P_2^{10}+P_2^{12}+P_2^{19})$$
$$+P_1^{14}(P_2^2+P_2^4+P_2^5+P_2^7)+P_1^{15}(P_2^4+P_2^6)]$$
$$+\frac{1}{(1-P_1^4)(1-P_1^6)(1-P_2^4)}[P_1(P_2^8+P_2^{11})$$
$$+P_1^2(P_2^4+P_2^6+P_2^8+P_2^{10}+P_2^{11}+P_2^{12}+P_2^{13}+P_2^{14})$$
$$+P_1^3(P_2^3+P_2^5+P_2^6+P_2^8+P_2^{13})$$
$$+P_1^4(P_2^7+P_2^8+P_2^9+P_2^{10}+P_2^{12}+P_2^{14}+P_2^{15})$$
$$+P_1^5(P_2^4+P_2^5+P_2^6+P_2^7+P_2^{10}+2P_2^{12}+P_2^{14})$$
$$+P_1^6(P_2^7+P_2^9+P_2^{10}+P_2^{12}+P_2^{15})$$
$$+P_1^7P_2^{14}+P_1^8(P_2^9+P_2^{11})+P_1^9P_2^{11}+P_1^{10}(P_2^3+P_2^{13})+P_1^{12}(P_2^3+P_2^{13})$$
$$+P_1^{13}P_2^5+P_1^{14}(P_2^5+P_2^7)+P_1^{15}P_2^2$$
$$+P_1^{16}(P_2+P_2^4+P_2^6+P_2^7+P_2^9)+P_1^{17}(P_2^2+2P_2^4+P_2^6+P_2^9+P_2^{10}+P_2^{12})$$
$$+P_1^{18}(P_2+P_2^2+P_2^4+P_2^6+P_2^7+P_2^8+P_2^9)$$
$$+P_1^{19}(P_2^3+P_2^8+P_2^{10}+P_2^{11})$$
$$+P_1^{20}(P_2^2+P_2^3+P_2^4+P_2^5+P_2^6+P_2^8+P_2^{10}+P_2^{12})$$
$$+P_1^{21}(P_2^5+P_2^8)]\Big)=1+P_1^4+P_2^4+\ldots. \qquad (65)$$

Here again an expansion of $\mathscr{F}(P_1, P_2)$ into power series yields the answer to the question about SU(3) representations containing the identity representation of $\Sigma(168)$. Namely, a term $mP_1^pP_2^q$ of the series implies that there are exactly m identity representations of $\Sigma(168)$ in the reduction of the representation (p, q) of SU(3).

The complexity of the expression (65) demonstrates how rich is the structure of integrity bases and their syzygies in the case of finite little groups of SU(3). A systematic study of these questions is being undertaken elsewhere (Desmier *et al* 1981). One cannot but wonder what implications, if any, could there be, for instance, for particle physics.

5. Concluding remarks

Many of the new generating functions of § 2, although fairly simple, cannot be obtained by a straightforward hand computation. The amount of algebraic manipulation involved (Patera and Sharp 1979) is prohibitive even if in some cases the newest techniques (Stanley 1980, King 1981, Baclawski 1981) could be used. The functions were therefore obtained either by going through a table of branching rules and compiling the integrity basis from there or from a direct examination of an algorithm defining the branching rule.

When a reference is given alongside a generating function it is to a paper which gives the generating function explicitly or, in some cases, gives either an integrity basis or a complete algorithm for the branching rule appropriate to all irreducible representations. If an integrity basis is completely determined, including polynomial identities (syzygies) relating its elements, the corresponding generating function can be written down relatively easily (Patera and Sharp 1980), and vice versa.

Most of the generating functions of § 2 refer to a maximal subgroup H of G. In physics, however, one is often interested in non-maximal little groups (O'Raifeartaigh 1979, Michel 1980). The corresponding generating function for subgroup scalars can be found by combining two or more generating functions for maximal subgroups (Patera and Sharp 1980). Let us exemplify the procedure by means of an example.

G(2) ⊃ SU(3) ⊃ SO(3)
The generating function (3) describes the first step. It has to be combined with the full generating function for the branching rules for G(2) ⊃ SU(3):

$$\frac{1}{(1-P_1P)(1-P_1Q)(1-P_2P)(1-P_2Q)}\left(\frac{1}{1-P_2}+\frac{P_1PQ}{1-P_1PQ}\right) \tag{66}$$

where the variables P_1 and P_2 carry the G(2) representation labels, and P and Q those of SU(3). From (3) it is clear that we need only that part of (66) which contains even degrees in P and Q. Separating that from (66) and setting $P = Q = 1$, we arrive at the desired generating function:

$$\frac{1}{(1-P_1^2)^2(1-P_2^2)^2}\left(\frac{(1+P_1P_2)^2}{1-P_2}+\frac{P_1(P_1+P_2)^2+P_1^2(1+P_1P_2)^2}{1-P_1^2}\right). \tag{67}$$

Another example, SU(3) ⊃ SO(3) ⊃ O, involving the octahedral group O, is given by equation (4.6) of Patera and Sharp (1980). There L should be replaced by unity and Λ_3 by zero. Then the coefficient of $\Lambda_1^p\Lambda_2^q$ is the number of O scalars in the SU(3) representation (p, q).

In all such cases where the subgroup H of G is not maximal it should be stressed that the subgroup scalar generating function does not automatically indicate that the subgroup in which the scalars occur is a little group. For example, the defining representation of SO($2k+1$) gives rise, as can be seen from (13), to a scalar of SU(k) × U(1). However the little group of this representation which leaves this scalar invariant is SO($2k$). Each such case of a non-maximal embedding must be treated with care in the determination of little groups.

Finally let us point out some further cases which could have been included in this paper.

Firstly the tables of branching rules of McKay and Patera (1981) and the results of Wybourne and Bowick (1977) and Wybourne (1978, 1979) could have been used to write down at least the first few terms in the expansions of the subgroup scalar generating functions for the maximal subgroups of the exceptional Lie groups.

Secondly the symmetries (45) and (46) could have been used along with the results of Sylvester (1881) and Sylvester and Franklin (1879) to extend the range of known generating functions $\mathscr{G}_{\{\mu\}}(L)$ to cover all partitions μ of 12 or less.

Thirdly the generating functions for finite subgroup scalars in representations of the Lorentz group O(3, 1) could be obtained from the work of Patera and Saint-Aubin (1980).

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